ELECTRIC, THERMAL, AND ELASTICITY FIELDS

IN A WIRE CONDUCTOR IN THE CASE OF A LOCAL JUMP IN RESISTANCE

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In the design of powerful electromagnetic systems it is necessary to calculate the thermomechanical processes that take place within such structures. Of particular interest are calculations of the fields that arise in the combined superconductors, so as to make possible an estimate of the undesirable consequences of a sudden transition to a normal state [1].

In the present study we examine processes in a single direct infinitely long combined wire conductor of circular cross section, representing a strand of an ideal conductor, coated with a thick layer of ordinary metal. It is assumed that the specific resistance of the strand undergoes a jumpwise change from zero to the level of the surrounding metal at the initial instant of time. In this case, the current which initially flows through this ideal conductor is expelled into the surrounding metal, heat is generated, and mechanical stresses are produced that can be attributed to the temperature difference and the ponderomotive forces.

This calculation is made up of three stages: the vector field of the current density is initially determined, and this is followed by the determination of the temperature field generated by the Joule heat; the results of these two stages are then used to calculate the mechanical stresses.

1. Current Density Diffusion. Using a system of Maxwell equations and the linear material relationships for an isotropic medium [2], we derive an equation for the current density

$$\frac{1}{\mu_0\mu\sigma}\nabla\times(\nabla\times\mathbf{j})+\frac{\partial\mathbf{j}}{\partial t}+\frac{\varepsilon_0\varepsilon}{\sigma}\frac{\partial^2\mathbf{j}}{\partial t^2}=0. \qquad (1.1)$$

Here $\mathbf{j}(r, t)$ is the current density vector as a function of the radius vector and of time; μ and ε denote the magnetic permeability and dielectric permittivity of the medium; μ_0 and ε_0 are the magnetic and electric constants; σ is the specific conductivity. Since we can neglect the displacement currents as small in comparison with the conductivity current in the case of metal conductors [2], we obtain from (1.1) $\Delta\mathbf{j} = \mu_0\mu\sigma\partial\mathbf{j}/\partial t$ (Δ is the Laplace operator).

In the problem which we are considering here, dealing with an infinitely long conductor, the vector $\mathbf{j}$ is directed along the axis, and with consideration of the symmetry of the circular cross section we arrive at the following:

$$\partial^2 j/\partial r^2+(1/r)\partial j/\partial r=\mu_0\mu\sigma\partial j/\partial t. \qquad (1.2)$$

We will assume that in the initial state the current I_0 is uniformly distributed over the cross section of a strand of radius e. The initial condition for Eq. (1.2) will then have the form

$$j(r,0)=\frac{I_0}{\pi e^2}\mathbf{1}(e-r), \qquad (1.3)$$

where $\mathbf{1}(\ldots)$ is the function of the single jump. The boundary conditions for (1.2) are represented by requirements of limits imposed on $j(0, t)$, as well as the relationship $\partial j(a, t)/\partial r = [\mu_0\mu\sigma/(2\pi a)](dI_0/dt)$, which is a result of the integration of (1.2) over the cross section of the conductor, i.e., a circle of radius a. Assuming subsequently the total current I_0 to be constant, we obtain $\partial j(a, t)/\partial r = 0$.

The formulated problem for the current density is solved by the Fourier method. As a result

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$$j(r,t)=\frac{I_0}{\pi a^2}+\frac{2I_0}{\pi e a}\sum_{n=1}^{\infty}\frac{J_1\left(\gamma_n\frac{e}{a}\right)J_0\left(\gamma_n\frac{r}{a}\right)}{\gamma_n J_0^2(\gamma_n)}\exp\left(-\frac{\gamma_n^2 t}{\mu_0\mu\sigma a^2}\right) \qquad (1.4)$$

(J_0, J_1 are the Bessel functions and γ_n denotes the roots of the equation $J_1(\gamma) = 0$).

For purposes of subsequent analytical calculation of the temperature and the mechanical stresses it is desirable to approximate (1.4) by means of some simple expression. For this we will use the variational equation [3]

$$\int_0^a (\Delta j-\mu_0\mu\sigma\partial j/\partial t)\,\delta j r\,dr=0, \qquad (1.5)$$

which is examined on a multiplicity of functions satisfying the condition $\partial j(a, t)/\partial r = 0$ and limited as $r \to 0$. We will specify the approximation

$$j_1=\frac{I_0}{\pi a^2}q_1(t)\left(1-\frac{r^2}{q_2^2(t)}\right)\mathbf{1}(q_2(t)-r). \qquad (1.6)$$

The varied functions q_1 and q_2 have the following sense: $q_1(t)$ is the value of the dimensionless current density at the center of the conductor and $q_2(t)$ is the coordinate of the diffusion front. The requirement of constancy for the total current $I_0=2\pi\int_0^a j_1 r\,dr$ associates q_1 with q_2: $q_2{}^2 = 3a^2/q_1$.

As a result of the substitution of (1.6) into (1.5), we obtain the following ordinary differential equation:

$$\frac{dq_1}{dt}+\frac{10}{3\mu_0\mu\sigma a^2}q_1^2=0. \qquad (1.7)$$

We will specify the initial condition for this equation so that the deviation norm $\int_0^a [j_1(r,0)-j(r,0)]^2 r\,dr$, where $j(r, 0)$ has the form of (1.3), is at its minimum. For a solution of (1.7) we will then have $q_1 = (10t/(3\mu_0\mu\sigma a^2) + 0.737e^2/a^2)^{-1}$, but it will be valid only under the condition that $q_2(t) \le a$, i.e., so long as the front of the current-density propagation does not reach the outside radius. This will occur at the instant of time $t_* = 0.1\mu_0\mu\sigma a^2(1 - 2.21e^2/a^2)$. When $t \ge t_*$ we will choose the approximation

$$j_2=\frac{I_0}{\pi a^2}\left[(q_3(t)-q_4(t))\left(1-\frac{r^2}{a^2}\right)^2+q_4(t)\right], \qquad (1.8)$$

for which we obtain

$$q_4=1-\exp\left(-\frac{15(t-t_*)}{\mu_0\mu\sigma a^2}\right),$$
$$q_3=3-2q_4.$$

Formulas (1.4), (1.6), and (1.8) have been used to carry out the calculations for a = 5 mm, e = 0.5 mm, $\mu = 1$, $\sigma = 5\cdot10^9$ $(\Omega\cdot m)^{-1}$ (copper with T = 18 K). The results of these calculations can be seen in Fig. 1 (the solid lines represent the precise solution, whereas the dashed lines are approximations).

2. Temperature Field. We assume that the wire conductor is surrounded by an infinitely isotropic uniform medium of low thermal conductivity. We will introduce the dimensionless radius $\rho = r/a$ and the time $\tau = t\kappa_1/(a^2C_1)$; κ_1 and κ_2 are the coefficients of thermal conductivity for the wire conductor and for the surrounding medium; C_1 and C_2 represent the heat capacities per unit volume for the wire conductor and the ambient medium.

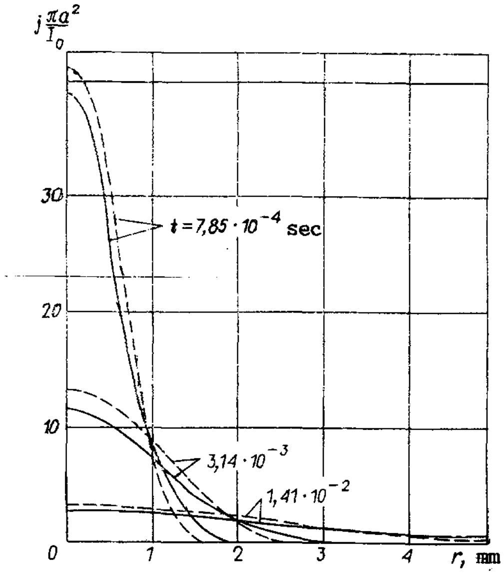

Fig. 1

The temperature field is described by the following system of equations:

$$\partial^2 T_1/\partial\rho^2 + (1/\rho)\partial T_1/\partial\rho + b = \partial T_1/\partial\tau,\ 0 < \rho < 1; \tag{2.1}$$

$$\lambda^2\partial^2 T_2/\partial\rho^2 = \eta^2\partial T_2/\partial\tau,\ \rho > 1, \tag{2.2}$$

where $\lambda^2 = \kappa_2/\kappa_1 \ll 1$; $\eta^2 = C_2/C_1$; we have introduced the Joule heat liberation $b = j^2a^2/(\sigma\kappa_1)$. In the notation of (2.2) we have made the assumption that the temperature field $T_2(\rho, \tau)$ in the ambient medium has the form of a boundary layer near $\rho = 1$, so that the term $\rho^{-1}\partial T_2/\partial\rho$ can be dropped.

The initial conditions $T_1(\rho, 0) = T_2(\rho, 0) = 0$ are associated with Eqs. (2.1) and (2.2), as well as the requirements of limitations on the solution as $\rho \to 0$ and $\rho \to \infty$, as well as the conjugacy conditions

$$T_1(1, \tau) = T_2(1, \tau) = \theta(\tau),\ \partial T_1(1, \tau)/\partial\rho = \lambda^2\partial T_2(1, \tau)/\partial\rho = Q(\tau).$$

Neither the temperature θ nor the dimensionless heat flow Q are known. However, applying the Laplace transform to Eq. (2.2), for the representation we obtain $\bar{\theta} = -(1/\lambda\eta\sqrt{p})Q$ (p is the Laplace transform parameter). Returning to the originals, we have

$$\theta(\tau) = -\frac{1}{\lambda\eta}\int_0^\tau \frac{Q(t)}{\sqrt{\pi(\tau - t)}}\,dt \tag{2.3}$$

which is a boundary condition for T_1. The subsequent calculation of the temperature is based on a solution of the integral equation for $Q(\tau)$, which is derived as follows. Assuming that $\partial T_1(1, \tau)/\partial\rho = Q$, we solve (2.1) by the method of eigenfunctions. Having substituted the found value of $T_1(1, \tau)$ into (2.3), we arrive at

$$2\int_0^\tau \left[1 + \sum_{n=1}^\infty \exp\left(-\gamma_n^2(\tau - t)\right)\right] Q(t)\,dt + \frac{1}{\lambda\eta}\int_0^\tau \frac{Q(t)\,dt}{\sqrt{\pi(\tau - t)}} = -F(\tau), \tag{2.4}$$

where

$$F(\tau) = 2\left[\int_0^\tau b_0(t)\,dt + \sum_{n=1}^\infty \frac{1}{J_0(\gamma_n)}\int_0^\tau \exp\left(-\gamma_n^2(\tau - t)\right) b_n(t)\,dt\right];$$

$$b_0 = \int_0^1 b(\rho, \tau)\,\rho\,d\rho, \quad b_n = \int_0^1 b(\rho, \tau)\,J_1(\gamma_n\rho)\,\rho\,d\rho;$$

as before, γ_n represents the roots of the equation $J_1(\gamma) = 0$.

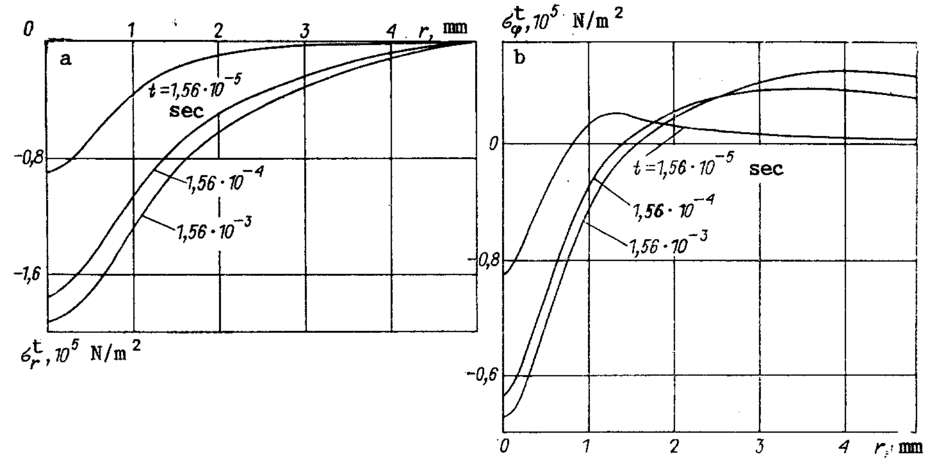

Fig. 2

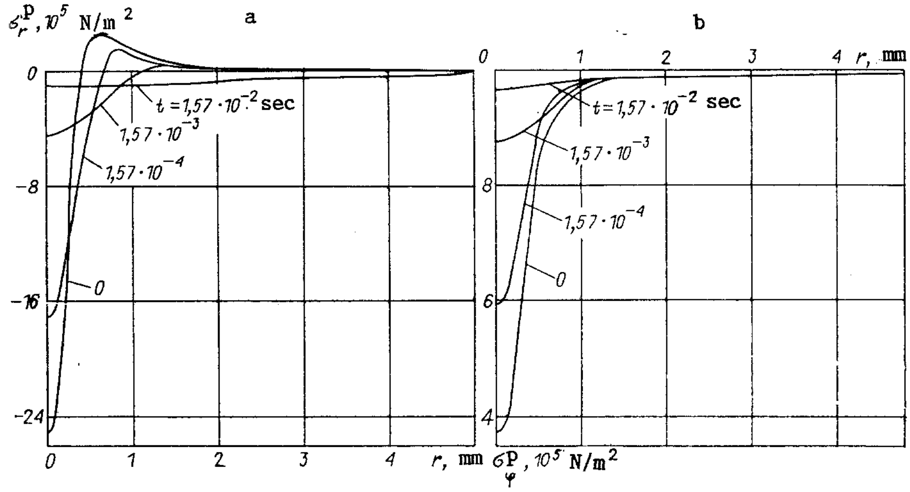

Fig. 3

The presence of the small parameter λ in Eq. (2.4) allows us to employ the asymptotic method. Looking for the solution in the form of $Q(\tau) = \lambda Q_1(\tau) + \lambda^2 Q_2(\tau) + \ldots$, we obtain

$$\int_0^\tau \frac{Q_1(t)}{\sqrt{\pi(\tau-t)}}\,dt = -\eta F(\tau),$$

$$\int_0^\tau \frac{Q_2(t)\,dt}{\sqrt{\pi(\tau-t)}} = -2\eta\int_0^\tau \left(1+\sum_{n=1}^{\infty}\exp\left(-\gamma_n^2(\tau-t)\right)Q_1(t)\right)dt = -\eta F_1(\tau)$$

etc. We have come to the Abel equations, whose solutions

$$Q_1 = -\eta\,\frac{d}{d\tau}\int_0^\tau \frac{F(t)}{\sqrt{\pi(\tau-t)}}\,dt$$

etc. When the evolution of heat is independent of time, we have

$$Q_1 = -\frac{4\eta}{\sqrt{\pi}}\left[b_0\sqrt{\tau}+\sum_{n=1}^{\infty}\frac{b_n}{J_0(\gamma_n)\gamma_n^2}\left(\sqrt{\tau}-\frac{\exp\left(-\gamma_n^2\tau\right)}{\gamma_n}\int_0^{\gamma_n\sqrt{\tau}}\exp(\lambda^2)\,d\lambda\right)\right].$$

There is no difficulty in calculating $Q_2(\tau)$.

Having determined the heat flow Q, we can find the sought temperature field T_1 in the conductor. As has been shown by the calculations, the process of heat conduction proceeds considerably more rapidly than the process of current diffusion. The greatest temperature differences arise in the initial period, when the current has not yet left the central strand. With the eigenfunction method we can obtain an expression for the temperature for this case

$$T_1(\rho, \tau) = 2b_0\tau + 4b_0 \frac{a}{e} \sum_{n=1}^{\infty} \frac{J_0(\gamma_n\rho) J_1(\gamma_n e)}{\gamma_n^3 J_0^2(\gamma_n)} (1 - \exp(-\gamma_n^2\tau)) \qquad (2.5)$$
$$(b_0 = (1/2)(I_0/(\pi e))^2/(\sigma\varkappa_1)).$$

3. Mechanical Stresses. The ponderomotive forces and the temperature differences lead to the appearance of mechanical stresses. Estimation of the frequencies of the elastic oscillations of the conductor in the case of axial symmetry showed that it is possible to limit ourselves to quasistatic calculations. The superposition principle valid for linear elasticity makes it possible independently to examine the effect of the temperature and electromagnetic field.

The system of equations from the theory of elasticity has the form [4]

$$\nabla\cdot\boldsymbol{\tau} + \mathbf{K} = 0, \ \boldsymbol{\varepsilon} = \nabla\mathbf{u}^s, \ \boldsymbol{\varepsilon} = (1/E)[(1+\nu)\boldsymbol{\tau} - \nu I_1(\boldsymbol{\tau})\mathbf{E}] + \alpha\mathbf{E}T,$$

where $\boldsymbol{\tau}$ is the stress tensor; $\mathbf{K}$ is the vector of the volumetric forces; $\boldsymbol{\varepsilon}$ is the strain tensor; $\mathbf{u}$ is the displacement vector; E is Young's modulus; ν is the Poisson coefficient; $I_1(\tau)$ is the first invariant of $\boldsymbol{\tau}$; $\mathbf{E}$ is the unit tensor; α is the temperature coefficient of linear expansion.

If we assume that the conductor is in a state of plane deformation, and if we take symmetry into consideration, for the radial component we find

$$r\sigma_r'' + 3r\sigma_r' + \frac{K_r}{1-\nu} + (rK_r)' + \frac{\alpha E}{1-\nu} T' = 0$$

$[(\ldots)' = d/dr]$. Assuming the surface $r = a$ to be free, we obtain

$$\sigma_r = \frac{1}{2(1-\nu)} \left[\left(\frac{1}{r^2} - \frac{1}{a^2}\right) \int_0^r r^2 K_r\, dr + \int_r^a K_r \left(1 - \frac{r^2}{a^2}\right) dr \right] +$$
$$+ \frac{1}{a^2} \int_0^a r \left(rK_r + \frac{\alpha E}{1-\nu} T\right) dr - \frac{1}{r^2} \int_0^r r \left(rK_r + \frac{\alpha E}{1-\nu} T\right) dr.$$

The remaining components of the stress tensor are determined with the equalities

$$\sigma_\varphi = (r\sigma_r)' + rK_r, \quad \sigma_z = \nu(\sigma_r + \sigma_\varphi) - \alpha ET.$$

For a conductor with $\mu = 1$ the only mechanical effect of the electromagnetic field is the Lorentz force [5]. In our case

$$K_r = -\mu_0 j(r, t) \frac{1}{r} \int_0^r jr\, dr, \quad K_\varphi = K_z = 0.$$

The calculations of the stresses were conducted with $\alpha = 10^{-6}$ K^{-1}, $\kappa_1 = 10^3$ W/(m·K), $E = 1.23\cdot10^{11}$ N/m^2, $\sigma = 0.5\cdot10^{10}$ $(\Omega\cdot\text{m})^{-1}$, $I_0 = 5\cdot10^3$ A.

The results from the calculation of the temperature stresses and the stresses due to the ponderomotive forces, at various instants of time, are shown, respectively, in Figs. 2 and 3. The resulting thermoelastic state of the conductor is not dangerous from the standpoint of strength. However, first of all, the parameters of the conductor have been chosen rather conditionally, and secondly, other criteria of operational capacity are possible (in the case of superconductors, for example, see [1]).

In the idealized formulation examined above we did not take into consideration the relationship between conductivity and other properties in their dependence on temperature. Consideration of this factor would have led to the associated nonlinear problem of current and temperature distribution. However, the cited results pertain to that initial stage of the process in which the heating is still small. According to formula (2.5), the temperature difference across the cross section amounts approximately to 2 K; with an initial temperature of 18 K the relationship between the properties and temperature in this case is not apparent [1].

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THE DYNAMICS OF AIR FLOW IN THE PRESENCE OF AN ENERGY PULSE IN THE SPHERICAL REGION, WITH PROVISION MADE FOR VIBRATIONAL—TRANSLATIONAL NONEQUILIBRIUM

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Numerous papers have been devoted to questions dealing with the dynamics of a gas in the presence of energy sources. We are thoroughly familiar with solutions to problems dealing with a powerful point explosion and with a spot explosion in which consideration is given to the counterpressure in an ideal gas with a constant adiabatic exponent for cases of plane, cylindrical, and spherical symmetry [1, 2]. Such explosions are similar to one another, if the variables are normalized to the corresponding parameters ρ_∞, p_∞ of the unperturbed gas, as well as to the characteristic dimensions and times of attenuation for the explosion wave, i.e., $r_0 = (E_0/p_\infty)^{1/n}$, $\tau_0 = r_0(\rho_\infty/p_\infty)^{1/2}$ (E_0 is the energy released per unit area or length, or the total energy of the explosion, n = 1, 2, 3 for plane, cylindrical, and spherical symmetry). The solution from point explosion theory (PET) frequently provides a good relationship for the magnitudes of the jumps in the gasdynamic variables at the front of the shock wave (SW) at great distances from the center of the explosion (when $r \gg R_0$, R_0 is the radius of the energy-release zone). However, in order to examine the distribution of the gasdynamic quantities over small periods of time, as well as to examine the finite distribution of temperature in the region of energy release after equalization of the pressure it is necessary to take into consideration the finiteness of the dimensions of the energy-release region and the time over which the energy contribution is effective.

The release of energy in a gas frequently comes about in nonequilibrium fashion. Thus, in a pulsed electric discharge in a molecular gas the greater portion of the released energy is stored in the vibrational degrees of molecular freedom, which leads to a significant divergence of the vibrational energy from equilibrium. In this case, in our analysis of the gasdynamic phenomena, we have to examine the kinetics of the exchange of energies between the internal and translational degrees of freedom for the molecules. The gasdynamics of nonequilibrium excited nitrogen was examined for instances of plane and cylindrical symmetry in [3, 4]. The duration of the excitation pulse was assumed, in this case, to be infinitely small.

The gasdynamic phenomena in the nonequilibrium excitation of the spherical region in air is examined in this study for various ratios of the time τ of the energy contribution and the characteristic gasdynamic and relaxation times.

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